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1 Basic Mensuration Formulas

These will be tested in 'Related Rates' and 'Optimisation' in Calculus. Know them!

**Cone**

\[
\text{Volume} = \frac{\pi r^2 h}{3} \\
\text{Surface Area} = \pi r^2 + \pi rl, \text{ where } l \text{ is the slanted height} \\
l^2 = h^2 + r^2
\]

**Cylinder**

\[
\text{Volume} = \pi r^2 h \\
\text{Surface Area} = 2\pi r^2 + 2\pi rh
\]

**Pyramid**

\[
\text{Volume} = \frac{\text{base area} \times \text{height}}{3}
\]

Note: The base will usually be a square, rectangle, or triangle in this course.

A tetrahedron is another name for triangular-based pyramid.
Sphere

Volume = \frac{4\pi r^3}{3}
Surface Area = 4\pi r^2

Cuboid

Volume = l \times w \times h
Surface Area = 2lw + 2wh + 2lh
2 Arc and Sector

- arc length $= r\theta$
- sector area $= \frac{1}{2}r^2\theta$

Questions may involve:

- find the angle, radius, arc length, sector area
- area of triangle using sine rule, isosceles triangle
- find the area of a segment.
- related rates with angle $\theta$ and its arc length, sector area, or area of segment
Figure 1 shows an arc $PQ$ of a circle with centre $O$.

The arc $PQ$ has length 20 cm.

The radius of the circle 12 cm.

The arc subtends angle $\theta$.

Find the shaded area.

\[ 12\theta = 20 \]
\[ \theta = \frac{20}{12} = \frac{5}{3} \]

sector area of $OPQ = \frac{1}{2}(12)^2 \left( \frac{5}{3} \right) = 120$

area of $\triangle OPQ = \frac{1}{2}(12)(12) \sin \left( \frac{5}{3} \right)$
\[ = 71.66937296 \]

shaded area = sector area of $OPQ$ – area of $\triangle OPQ$
\[ = 120 - 71.66937296 \]
\[ \approx 48.3 \text{ cm (3 s.f.)} \]
3 Linear Programming

For linear inequalities,

- set \( x = 0 \) to get \( y \)-coordinate, which will be the \( y \)-intercept. \((0, y)\)

- set \( y = 0 \) to get \( x \)-coordinate, which will be the \( x \)-intercept. \((x, 0)\)

e.g. \( 3x + 2y = 5 \)
Set \( x = 0 \),
\[
3(0) + 2y = 55
\]
\[
2y = 55
\]
\[
y = \frac{55}{2} \rightarrow (0, \frac{55}{2}) \text{ \( x \)-intercept}
\]
Set \( y = 0 \),
\[
3x + 2(0) = 5
\]
\[
3x = 5
\]
\[
x = \frac{5}{3} \rightarrow (\frac{5}{3}, 0) \text{ \( y \)-intercept}
\]

Join these two points with a line and extend to the border of the grid provided.

If the intersecting points NOT on the lattice points (integer coordinates), solve the equations to find the exact coordinates.
4 Quadratic Inequalities

If \( x^2 > p \), then \( x > \sqrt{p} \) or \( x < -\sqrt{p} \)

If \( x^2 < p \), then \( -\sqrt{p} < x < \sqrt{p} \)

If \((x - a)(x - b) > p\), then \( x < a \) or \( x > b \), \( a > b \)

If \((x - a)(x - b) < p\) then \( a < x < b \)

In the above cases, \( a \) is less than \( b \). If not, switch values around so that they are.

e.g. Find the range of values for \( x \) in which \( x^2 - 2x - 3 \geq 0 \)

\[
x^2 - 2x - 3 \geq 0
\]
\[
(x - 3)(x + 1) \geq 0
\]
critical values = 3, -1

\[
x \leq -1 \text{ or } x \geq 3
\]

You can sketch the curve or use a sign diagram. (Use a GDC to confirm if you have time.)

You don’t have to draw both, I am just showing you their relationships.
5 Discriminant

Given $ax^2 + bx + c$, the discriminant, denoted by the symbol $\Delta$ (delta), is

$$\Delta = b^2 - 4ac$$

If a quadratic has two (different, distinct) real roots, $\Delta > 0$.
If a quadratic has equal roots, $\Delta = 0$
If a quadratic has no real roots, $\Delta < 0$.

e.g. Find the set of values of $p$ for which $2x^2 - px + 3$ has no real roots.

$$\Delta = b^2 - 4ac$$
$$= (-p)^2 - 4(2)(3)$$
$$= p^2 - 24$$

Set $\Delta < 0$,

$$p^2 - 24 < 0$$

$$p^2 < 24$$

$$|p| < \sqrt{24}$$

$$|p| < 2\sqrt{6}$$

$$-2\sqrt{6} < p < 2\sqrt{6}$$
6 Completing the Square

Rewrite equation in **DESCENDING** powers first, factorise the first coefficient out of the first two terms, half the linear term, square it and subtract it. Distribute and collect the constant terms.

e.g. $f(x) = 2x^2 - 5x + 8$

Find the coordinates of the vertex, state whether it is a maximum or minimum.

\[
f(x) = 2x^2 - 5x + 8 \\
= 2\left(x^2 - \frac{5}{2}x\right) + 8 \\
= 2\left(x^2 + 2x\left(-\frac{5}{4}\right) + \left(-\frac{5}{4}\right)^2 - \left(-\frac{5}{4}\right)^2\right) + 8 \\
= 2\left\{\left(x - \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right\} + 8 \\
= 2\left\{\left(x - \frac{5}{4}\right)^2 - \frac{25}{16}\right\} + 8 \\
= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} + 8 \\
= 2\left(x - \frac{5}{4}\right)^2 + \frac{39}{8}
\]

Recall: $x^2 + 2xy + y^2 = (x+y)^2$

Note the extra set of brackets

The vertex is $\left(\frac{5}{4}, \frac{39}{8}\right)$. It is a minimum because the coefficient of $x^2$ ($=2$) is positive.

In other words, the $x$-coordinate for which this minimum occurs is the $x$-coordinate, $x = \frac{5}{4}$.

The minimum value is just the constant term $\frac{39}{8}$.

To write out, I would skip some of the steps above.

\[
f(x) = 2x^2 - 5x + 8 \\
= 2\left(x^2 - \frac{5}{2}x\right) + 8 \\
= 2\left\{\left(x - \frac{5}{4}\right)^2 - \frac{25}{16}\right\} + 8 \\
= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} + 8 \\
= 2\left(x - \frac{5}{4}\right)^2 + \frac{39}{8}
\]
e.g. \( g(x) = 6 - 3x - 4x^2 \)

(i) Find the value of \( x \) for which \( g(x) \) has a maximum,

(ii) Find the maximum value.

\[
g(x) = -4x^2 - 3x + 6 \quad \leftrightarrow \text{reorder in descending powers}
\]

\[
g(x) = -4 \left( x^2 + \frac{3}{4}x \right) + 6
\]

\[
= -4 \left( x^2 + 2x \left( \frac{3}{8} \right) + \left( \frac{3}{8} \right)^2 - \left( \frac{3}{8} \right)^2 \right) + 6
\]

\[
= -4 \left[ \left( x + \frac{3}{8} \right)^2 - \left( \frac{3}{8} \right)^2 \right] + 6
\]

\[
= -4 \left[ \left( x + \frac{3}{8} \right)^2 - \frac{9}{64} \right] + 6
\]

\[
= -4 \left( x + \frac{3}{8} \right)^2 + \frac{9}{16} + 6 \quad \text{note the change of signs}
\]

\[
= -4 \left( x + \frac{3}{8} \right)^2 + \frac{105}{16}
\]

(i) \( x = -\frac{3}{8} \)

(ii) \( \frac{105}{16} \)

Again, to save time.

\[
g(x) = -4x^2 - 3x + 6
\]

\[
= -4 \left( x^2 + \frac{3}{4}x \right) + 6
\]

\[
= -4 \left[ \left( x + \frac{3}{8} \right)^2 - \frac{9}{64} \right] + 6
\]

\[
= -4 \left( x + \frac{3}{8} \right)^2 + \frac{9}{16} + 6
\]

\[
= -4 \left( x + \frac{3}{8} \right)^2 + \frac{105}{16}
\]
6.1 Axis of Symmetry

Each quadratic equation is symmetric at \( x = \frac{-b}{2a} \). This is the result of completing the square! When \( x = \frac{-b}{2a} \), the perfect square will tend to zero, and the remaining term must be the minimum or maximum value.

Using the two examples above,
\[
f(x) = 2x^2 - 5x + 8
\]
\[
x = \frac{-(-5)}{2(2)} = \frac{5}{4}
\]
\[
f \left( \frac{5}{4} \right) = 2 \left( \frac{5}{4} \right)^2 - 5 \left( \frac{5}{4} \right) + 8 = \frac{39}{8}
\]

\[
g(x) = 6 - 3x - 4x^2
\]
\[
= -4x^2 - 3x + 6
\]
I still recommend rearranging the powers in descending order, so you don’t erroneously pick out the wrong coefficients of \( a, b, \) and \( c \).
\[
x = \frac{-(-3)}{2(-4)} = \frac{3}{-8}
\]
\[
g \left( \frac{3}{-8} \right) = 6 - 3 \left( \frac{3}{-8} \right) - 4 \left( \frac{3}{-8} \right)^2 = \frac{105}{16}
\]
We can use the results and combine it back into the form \( a(x - h)^2 + k \).

6.2 Differentiation

You can differentiate to find the maximum/minimum.
\[
f(x) = 2x^2 - 5x + 8
\]
\[
f'(x) = 4x - 5
\]
Set \( f'(x) = 0, \)
\[
4x - 5 = 0
\]
\[
x = \frac{5}{4}
\]
\[
f \left( \frac{5}{2} \right) = 2 \left( \frac{5}{2} \right)^2 - 5 \left( \frac{5}{2} \right) + 8 = \frac{39}{8}
\]
\[
g(x) = -4x^2 - 3x + 6
\]
\[
g'(x) = -8x - 3
\]
Set \( g'(x) = 0 \)

\[-8x - 3 = 0\]

\[x = \frac{3}{-8}\]

\[g\left( \frac{3}{-8} \right) = 6 - 3 \left( \frac{3}{-8} \right) - 4 \left( \frac{3}{-8} \right)^2 = \frac{105}{16}\]

### 6.3 Recommended Read

7 Basic Differentiation

Basic Table for Differentiation

\[
\begin{align*}
\frac{d}{dx}[x^n] &= nx^{n-1} \\
\frac{d}{dx}\left[\frac{1}{x^n}\right] &= \frac{-n}{x^{n+1}} \\
\frac{d}{dx}[e^x] &= e^x \\
\frac{d}{dx}[e^{ax}] &= ae^{ax} \\
\frac{d}{dx}[\sin x] &= \cos x \\
\frac{d}{dx}[\sin ax] &= a \cos ax \\
\frac{d}{dx}[\cos x] &= -\sin x \\
\frac{d}{dx}[\cos ax] &= -a \sin ax
\end{align*}
\]

7.1 Product Rule

\[
\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)
\]

e.g. Differentiate \(3x^3 \cos 2x\).

\[
f(x) = 3x^3 \quad g(x) = \cos 2x
\]

\[
f'(x) = 9x^2 \quad g'(x) = -2 \sin 2x
\]

\[
\frac{d}{dx}[3x^3 \cos 2x] = 9x^2(\cos 2x) + 3x^3(-2 \sin 2x)
\]

\[
= 9x^2 \cos 2x - 6x^3 \sin 2x
\]

7.2 Quotient Rule

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}
\]

e.g. Differentiate \(\frac{e^{2x}}{x^3}\).

\[
f(x) = e^{2x} \quad g(x) = x^3
\]

\[
f'(x) = 2e^{2x} \quad g'(x) = 3x^2
\]

\[
\frac{d}{dx} \left[ \frac{e^{2x}}{x^3} \right] = \frac{2e^{2x}(x^3) - e^{2x}(3x^2)}{[x^3]^2}
\]

\[
= \frac{2x^3e^{2x} - 3x^2e^{2x}}{x^6} = \frac{e^{2x}(2x^3 - 3x^2)}{x^6}
\]

factorisation is optional here*

*Last revised on 13/5/2018 by Steve Cheung
*You will find it helpful to factorise if this equation is to be used in later steps, such as finding the second derivative or solving for 0.

7.3 Chain Rule

\[
\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g(x)
\]

or

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}
\]

This can be further extended to:

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \ldots \times \frac{dw}{dx}
\]
8 Solve by Graphing

- Use a calculator to substitute values, highly recommend storing $X$ as a variable and replace
  the value of $X$ to reduce human error.

- I find it easier to change the target equation back to the reference curve. But it is entirely
  possible to go the other way around.

- Check your final answer by plugging it into the target equation, the left-hand-side and right-
  hand side should be really close. Not only that, by check one unit of accuracy to the left and
  the right and make sure you answer is the closest.
9 Arithmetic Sequences and Series

\[ n \text{th term: } t_n = a + (n - 1)d \]
\[ t_1 = a \]
\[ t_2 = a + d \]
\[ t_3 = a + 2d \]
\[ t_4 = a + 3d \]
\[ t_5 = a + 4d \]
\[ \vdots \]

\( a \) is the first term, \((t_1)\); \( l \) is the last term \((t_n)\); \( d \) is the common difference

\[ S_n = \frac{n}{2}[a + l] \]

\[ = \frac{n}{2}[2a + (n - 1)d] \quad \text{replace } l = a + (n - 1)d \]

To find the common difference, it usually requires you to set two expressions in the form of \( a + (n - 1)d \)
then form a linear equation and solve. This could potentially lead to solving simultaneous equations
in linear.

e.g. In an arithmetic sequence, the third term is four times the sixth form. The fifth term is 42.

Find the first term and the common difference.

Solution:

\[ t_3 = 4t_6 \] \hspace{1cm} \( (t_5 = a + 4d) \) \hspace{1cm} \( a = -6d \)
\[ a + 2d = 4(a + 5d) \] \hspace{1cm} \( a + 4d = 42 \) \hspace{1cm} \( = -6(-21) \)
\[ a + 2d = 4a + 20d \] \hspace{1cm} \( -6d + 4d = 42 \) \hspace{1cm} \( = 126 \)
\[ -3a = 18d \] \hspace{1cm} \( -2d = 42 \)
\[ a = -6d \] \hspace{1cm} \( d = -21 \)
10 Geometric Sequences and Series

nth term: \( t_n = ar^{n-1} \)

\( t_1 = a \)
\( t_2 = ar \)
\( t_3 = ar^2 \)
\( t_4 = ar^3 \)
\( \vdots \)

\( a \) is the first term, \((t_1)\); \( r \) is the common ratio

\[ S_n = \frac{a(1 - r^n)}{1 - r} \] (finite number of terms)

\[ S_\infty = \frac{a}{1 - r}, |r| < 1 \] (infinite number of terms)

\( r \neq 0 \) for both summation formulas above.

Similar to arithmetic sequences, to find the common ratio, it usually requires you to set two expressions in the form of \( ar^{n-1} \), then form an equation and solve by factorisation. You may also divide powers of \( r \) for this type of question because \( r \) is presumed to not equal to 0. This could potentially lead to solving simultaneous equations in non-linear, and the trick is to divide the equations.
In a geometric series, the sixth term is three times the second term. The fourth term is 30.

(a) Find the common ratio of the sequence.

(b) The first term of this geometric series is \(a\). State the range of \(a\) for which this geometric series has an infinite sum.

Solution:

(a) \(t_6 = 3t_2\) \(t_4 = 30\)
\[
ar^5 = 3ar \quad ar^3 = 30
\]
\[
\frac{ar^5}{ar^3} = \frac{3ar}{30}
\]
\[
r^2 = \frac{ar}{10}
\]
\[
r = \frac{a}{10}
\]

(b) \(|r| < 1\)
\[
\left| \frac{a}{10} \right| < 1
\]
\[
|a| < 10
\]
e.g. In a geometric series, the sum of the second and fourth term is equal to the sixth term.

(a) Find the common ratio.

(b) Given that the first term is 4, \(r > 0\). Find the least value of \(n\) for which \(S_n > 760\).

Solution:

(a) \(t_2 + t_4 = t_6\)
\[
ar + ar^3 = ar^5
\]
\[
ar(1 + r^2) = ar^5
\]
\[
1 + r^2 = r^4
\]
\[
0 = r^4 - r^2 - 1
\]
\[
r^2 = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}
\]
\[
r^2 = \frac{1 \pm \sqrt{5}}{2}
\]
\[
r = \pm \sqrt{\frac{1 \pm \sqrt{5}}{2}}
\]
(b) \( S_n > 760 \)

\[ \frac{a(1 - r^n)}{1 - r} > 760 \]

\[ 4 \left( 1 - \left( \frac{1 + \sqrt{5}}{2} \right)^n \right) > 760 \]

\[ 1 - \sqrt{\frac{1 + \sqrt{5}}{2}} \]

\[ 1 - \left( \frac{1 + \sqrt{5}}{2} \right)^n < \frac{760}{4} \left( 1 - \sqrt{\frac{1 + \sqrt{5}}{2}} \right) < 190 \left( 1 - \sqrt{\frac{1 + \sqrt{5}}{2}} \right) - 1 \]

\[ \left( \frac{1 + \sqrt{5}}{2} \right)^n > 1 - 190 \left( 1 - \sqrt{\frac{1 + \sqrt{5}}{2}} \right) \]

\[ \log \left( \frac{1 + \sqrt{5}}{2} \right)^n > \log \left( 1 - 190 \left( 1 - \sqrt{\frac{1 + \sqrt{5}}{2}} \right) \right) > \log \left( 1 - 190 \left( 1 - \sqrt{\frac{1 + \sqrt{5}}{2}} \right) \right) \]

\[ n \log \left( \frac{1 + \sqrt{5}}{2} \right) > \log \left( 1 - 190 \left( 1 - \sqrt{\frac{1 + \sqrt{5}}{2}} \right) \right) \]

\[ n > \frac{\log \left( 1 - 190 \left( 1 - \sqrt{\frac{1 + \sqrt{5}}{2}} \right) \right)}{\log \left( \frac{1 + \sqrt{5}}{2} \right)} \]

\[ n > 16.47634799 \]

\[ n = 17 \]

Notice that \( n \) follows the form:

\[ n > \frac{\log \left( 1 - \frac{s}{a}(1 - r) \right)}{\log r} \]

where \( s \) is the bound of the sum.

This is true for \( r > 0 \) but not for \( r < 0 \).
11 Summation, Sigma Notation

Summations in this course can usually be thought as arithmetic series. The easiest approach is to evaluate the arithmetic sum using the first and the last term. Count the number of terms carefully!

\[ \sum_{r=i}^{j} (pr + q) = \frac{j - i + 1}{2} (pi + q + pj + q) \]

I do not suggest memorising this. Breaking it down in parts is easier.

e.g. Evaluate \( \sum_{r=12}^{22} (2r - 3) \).

It is a good idea to write down the first few terms to confirm that it is an arithmetic series first.

\[ \sum_{r=12}^{22} (2r - 3) = (2(12) - 3) + (2(13) - 3) + (2(14) - 3) + ... + (2(22) - 3) \]

\[ = 21 + 23 + 25 + ... + 41 \]

\( a = 2(12) - 3 = 21 \)

\( l = 2(22) - 3 = 41 \)

\( n = 22 - 12 + 1 = 11 \) (remember to add 1!)

\[ \sum_{r=12}^{22} (2r - 3) = \frac{n}{2} (a + l) \]

\[ = \frac{11}{2} (21 + 41) \]

\[ = \frac{11}{2} (62) \]

\[ = 341 \]
e.g. Evaluate $\sum_{r=1}^{20} 2(3)^r$.

$$\sum_{r=1}^{20} 2(3)^r = 2(3) + 2(3)^2 + 2(3)^3 ... + 2(3)^{20}$$

$$= 6 + 18 + 54 + ...$$

This is actually a geometric series!

$a = 6$

$r = 3$

$$n = 20 - 1 + 1 = 20$$

$$\sum_{r=1}^{20} 2(3)^r = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{6(1 - 3^{20})}{1 - 3}$$

$$\approx 6.18 \times 10^{14}$$

Given that $\sum_{r=4}^{m} (5r + 6) = 2618$, solve for $m$.

$a = 5(4) + 6 = 26$

$l = 5m + 6$

$$n = m - 4 + 1 = m - 3$$

$$\sum_{r=4}^{m} (5r + 6) = 2618$$

$$\frac{n}{2}(a + l) = 2618$$

$$\frac{m - 3}{2}(26 + (5m + 6)) = 2618$$

$$(m - 3)(5m + 32) = 5236$$

$5m^2 + 32m - 15m - 96 = 5236$

$5m^2 + 17m - 96 = 5236$

$5m^2 + 17m - 5332 = 0$

$$m = \frac{-17 \pm \sqrt{(-17)^2 - 4(5)(-5332)}}{2(5)}$$

$$m = \frac{-17 \pm 327}{10}$$

$$m = 31 \text{ or } \frac{-172}{5} \text{ (reject)}$$

$$m = 31$$
12 Alpha Beta, Vieta’s Formulas

Given a quadratic in the form \( ax^2 + bx + c = 0 \), the two roots \( \alpha \), and \( \beta \), which come from the factored form \( a(x - \alpha)(x - \beta) = 0 \), form the following relations:

\[
\begin{align*}
\bullet & \quad \alpha + \beta = -\frac{b}{a} \\
\bullet & \quad \alpha \beta = \frac{c}{a}
\end{align*}
\]

Proof:
\[
a(x - \alpha)(x - \beta) = a(x^2 - x\beta - x\alpha + \alpha\beta)
= a(x^2 - x(\beta + \alpha) + \alpha\beta)
= a(x^2 - (\alpha + \beta)x + \alpha\beta)
\]
Equate the two quadratic forms:
\[
a(x^2 - (\alpha + \beta)x + \alpha\beta) = ax^2 + bx + c
\]
\[
x^2 - (\alpha + \beta)x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}
\]
Matching the coefficients, we get:
\[
-(\alpha + \beta) = \frac{b}{a} \Rightarrow \alpha + \beta = -\frac{b}{a}
\]
\[
\alpha + \beta = \frac{c}{a}
\]
\[\square\]

The following combinations of the roots are often tested:

\bullet \quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta

\bullet \quad \alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta) = (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]

\bullet \quad \alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2) = (\alpha - \beta)(\alpha^2 + \beta^2 + \alpha\beta) = (\alpha - \beta)[(\alpha + \beta)^2 - \alpha\beta]

\bullet \quad \alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha^2\beta^2) = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2

\bullet \quad \alpha - \beta = \pm \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}

\quad \text{– if } \alpha > \beta, \text{ then } \alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}

\quad \text{– if } \alpha < \beta, \text{ then } \alpha - \beta = -\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}

* \quad \alpha^4 - \beta^4 = (\alpha^2 + \beta^2)(\alpha^2 - \beta^2) = (\alpha^2 + \beta^2)(\alpha + \beta)(\alpha - \beta)

\quad = \pm [(\alpha + \beta)^2 - 2\alpha\beta](\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}

I don’t recommend memorising the final form. \( \alpha - \beta \) should be given to use this.
I personally prefer the underlined versions, but it would require you to find the sum of squares \((\alpha^2 + \beta^2)\) first. This will often be asked in a previous part anyway. If not, you find it and write on the top of the page for easy access.

♀Tip: Write the values of \(\alpha + \beta\), \(\alpha\beta\), and \(\alpha^2 + \beta^2\) at the beginning, even if they are given. You will always need to refer back to them as you do calculations.
13 Simultaneous Equations

Always substitute a linear equation into a non-linear one. Sometimes a change of variable (substitution) is required to change it into a recognisable form.
14 **Rational Function** \( f(x) = \frac{ax + b}{cx + d} \)

- horizontal asymptote: substitute a very large number for \( x \), say, 999 999.
- vertical asymptote: set the denominator equal to zero, and solve for \( x \).

Potential problems may involve the following:

- find the coordinates where the graph crosses the axes \( \Rightarrow \) find \( x \) and \( y \)-intercepts

- find the equation of the tangent line at a given \( x \)-coordinate \( (x = p) \) or point \((p, q)\) where \( q = f(a) \)

  1. find \( \frac{dy}{dx} \bigg|_{x=p} \), this will be \( m \) (requires using ‘quotient rule’)
  2. use \( y - q = m(x - p) \) and rearrange

- find the equation of the normal \( \Rightarrow \) replace \( m \) with \( \frac{-1}{m} \) in step 1 above

- use the tangent or normal to find where it crosses the curve once again, potentially leading the solving a quadratic equation
15 Integration

15.1 Indefinite Integrals

\[ \int f(x) \, dx = F(x) + C \] needs a constant (of integration) \( C \) at the end of the expression

<table>
<thead>
<tr>
<th>Basic Table for Integration</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int x^n , dx = \frac{x^{n+1}}{n+1} + C )</td>
</tr>
<tr>
<td>( \int e^x , dx = e^x + C )</td>
</tr>
<tr>
<td>( \int e^{ax} , dx = \frac{e^{ax}}{a} + C )</td>
</tr>
<tr>
<td>( \int \sin x , dx = -\cos x + C )</td>
</tr>
<tr>
<td>( \int \sin ax , dx = -\frac{\cos ax}{a} + C )</td>
</tr>
<tr>
<td>( \int \cos x , dx = \sin x + C )</td>
</tr>
<tr>
<td>( \int \cos ax , dx = \frac{\sin ax}{a} + C )</td>
</tr>
</tbody>
</table>

You should differentiate your answer to see if it gets back to the integrand, then you would know if it is correct or not, provided that you write down the constant \( C \).

e.g. Find \( \int (x^3 + \frac{1}{2}x^2 + 6) \, dx \).

\[
\int (x^3 + \frac{1}{2}x^2 + 6) \, dx = \frac{x^4}{4} + \frac{1}{2} \left( \frac{x^3}{3} \right) + 6x + C
\]

\[
= \frac{x^4}{4} + \frac{x^3}{6} + 6x + C
\]

Check answer

\[
\frac{d}{dx} \left( \frac{x^4}{4} + \frac{x^3}{6} + 6x + C \right) = \frac{4x^3}{4} + \frac{3x^2}{6} + 6 + 0
\]

\[
= x^3 + \frac{x^2}{2} + 6 \quad \checkmark
\]

15.2 Definite Integrals

\[
\int_a^b f(x) \, dx = \left[ F(x) + C \right]_a^b = \left[ F(x) \right]_a^b + \int_a^b C \, dx = \left[ F(x) \right]_a^b = F(b) - F(a)
\]
e.g. Evaluate \( \int_0^{\pi} 4 \sin 2\theta \, d\theta \).

\[
\int_0^{\pi} 4 \sin 2\theta \, d\theta = \left[ \frac{-4 \cos 2\theta}{2} \right]_0^{\pi} = \left[ -2 \cos 2\theta \right]_0^{\pi} = \left( -2 \cos 2 \left(\frac{\pi}{2}\right) \right) - \left( -2 \cos 2 \left(0\right) \right) = (-2 \cos \pi) - (-2 \cos 0) = (-2 \times -1) - (-2 \times 1) = 2 - (-2) = 4
\]

e.g. Evaluate \( \int_0^3 (2x^3 - 3x^2 + x - 6) \, dx \).

\[
\int_0^3 (2x^3 - 3x^2 + x - 6) = \left[ \frac{2x^4}{4} - \frac{3x^3}{3} + \frac{x^2}{2} - 6x \right]_0^3 = \left[ \frac{x^4}{2} - x^3 + \frac{x^2}{2} - 6x \right]_0^3 = \left( \frac{(3)^4}{2} - (3)^3 + \frac{(3)^2}{2} - 6(3) \right) - \left( \frac{(0)^4}{2} - (0)^3 + \frac{(0)^2}{2} - 6(0) \right) = 0 - 0 = 0
\]

\[\textbf{\textcolor{purple}{Tip:}} \text{ When dealing with polynomials, you can save time when you substitute 0 as one of your limits. You can replace the (\*) line with}
\]

\[
= \left( \frac{(3)^4}{2} - (3)^3 + \frac{(3)^2}{2} - 6(3) \right) - 0
\]

I also recommend substituting your values with an expression in your calculator, store the limits as \( x \) values, and run it through the expression. Check your answer by integrating directly using your calculator.

\[\textbf{\textcolor{purple}{Note:}} \text{ Polynomials are functions like } x^3 + 5x^2 - \frac{1}{2}x, \text{ } 2x^2 - 5. \]

Don’t do it for \( \sin x, \cos x, \tan x, e^x, \ln x, \log x, \) and the like! (See the previous example.)
16  Binomial Expansion

When $n$ is a natural number, $n = 1, 2, 3, 4, ...$,

$$(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

$= (\binom{n}{0}) a^n b^0 + (\binom{n}{1}) a^{n-1} b^1 + (\binom{n}{2}) a^{n-2} b^2 + ... + (\binom{n}{n-1}) a^1 b^{n-1} + (\binom{n}{n}) a^0 b^n$

$= a^n + (\binom{n}{1}) a^{n-1} b + (\binom{n}{2}) a^{n-2} b^2 + ... + (\binom{n}{n-1}) a b^{n-1} + b^n$

$$(a + b)^n = a^n + (\binom{n}{1}) a^{n-1} b + (\binom{n}{2}) a^{n-2} b^2 + ... + (\binom{n}{n-1}) a b^{n-1} + b^n$$

$$(1 + px)^n = 1^n + (\binom{n}{1}) 1^{n-1} (px) + (\binom{n}{2}) 1^{n-2} (px)^2 + ... + (\binom{n}{n-1}) 1 (px)^{n-1} + (px)^n$$

$= 1 + (\binom{n}{1}) px + (\binom{n}{2}) (px)^2 + ... + (\binom{n}{n-1}) (px)^{n-1} + (px)^n$

$$(1 + px)^n = 1 + (\binom{n}{1}) px + (\binom{n}{2}) (px)^2 + ... + (\binom{n}{n-1}) (px)^{n-1} + (px)^n$$

e.g. Expand $(3x + 4)^5$,

$$(3x - 4)^5 = (3x)^5 + \binom{5}{1} (3x)^4 (-4) + \binom{5}{2} (3x)^3 (-4)^2 + \binom{5}{3} (3x)^2 (-4)^3 + \binom{5}{4} (3x)^1 (-4)^4 + (-4)^5$$

$= 243x^5 - 1620x^4 + 4320x^3 - 5760x^2 + 3840x - 1024$

e.g. Expand $\left(1 + \frac{1}{3} x\right)^4$.

$$\left(1 + \frac{1}{3} x\right)^4 = 1 + (\binom{4}{1} \left(\frac{1}{3} x\right)) + (\binom{4}{2} \left(\frac{1}{3} x\right)^2) + (\binom{4}{3} \left(\frac{1}{3} x\right)^3) + (\binom{4}{4} \left(\frac{1}{3} x\right)^4)$$

$= 1 + \frac{4}{3} x + \frac{2}{3} x^2 + \frac{4}{27} x^3 + \frac{1}{81} x^4$

When $n$ is not a natural number, $|px| < 1$ (e.g. $n = -2, \frac{1}{2}, -\frac{1}{3}, 0.2, \pi$)

$$(1 + px)^n = 1 + (\binom{n}{1}) px + (\binom{n}{2}) (px)^2 + (\binom{n}{3}) (px)^3 + ...$$

$= 1 + npx + \frac{n(n-1)}{2!} (px)^2 + \frac{n(n-1)(n-2)}{3!} (px)^3 + ...$

$$(1 + px)^n = 1 + npx + \frac{n(n-1)}{2!} (px)^2 + \frac{n(n-1)(n-2)}{3!} (px)^3 + ... \quad |x| < \frac{1}{|p|}$$

Sometimes, this requires factorisation in order to be used.

$$(a + bx)^n = a^n \left(1 + \frac{b}{a} x\right)^n = a^n \left(1 + n \left(\frac{b}{a} x\right) + \frac{n(n-1)}{2!} \left(\frac{b}{a} x\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{b}{a} x\right)^3 + ...\right)$$
e.g. Expand and give a simplified expression for $\sqrt{1-5x}$ up to and including the term $x^3$. Write down the values of $x$ in which this expansion is valid.

$$\sqrt{1-5x} = (1-5x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-5x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-5x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-5x)^3 + ...$$

$$= 1 - \frac{5}{2}x - \frac{25}{8}x^2 - \frac{125}{16}x^3 + ...$$

$$|x| < \frac{1}{5}$$

e.g. Expand and give a simplified expression for $\frac{1}{4x^2 + 1}$ up to and including the term $x^4$. Write down the values of $x$ in which this expansion is valid.

$$\frac{1}{4x^2 + 1} = (4x^2 + 1)^{-1}$$

$$= (1 + 4x^2)^{-1}$$

$$= 1 + (-1)(4x^2) + \frac{(-1)(-2)}{2!}(4x^2)^2 + ...$$

$$= 1 - 4x^2 + 16^4 + ...$$

$$|x^2| < \frac{1}{4}$$

$$x^2 < \frac{1}{4}$$

$$|x| < \sqrt{\frac{1}{4}}$$

$$|x| < \frac{1}{2}$$

e.g. Expand $\frac{\sqrt{1-5x}}{4x^2 + 1}$ in ascending powers, up to and including the term $x^3$. Write down the values of $x$ in which this expansion is valid

$$\frac{\sqrt{1-5x}}{4x^2 + 1} = \left(1 - \frac{5}{2}x - \frac{25}{8}x^2 - \frac{125}{16}x^3 + ...\right)\left(1 - 4x^2 + 8x^4 + ...\right)$$

$$= 1 - \frac{5}{2}x - \frac{25}{8}x^2 - \frac{125}{16}x^3 - 4x^2 + 10x^3 + ...$$

$$= 1 - \frac{5}{2}x - \frac{57}{8}x^2 + \frac{35}{16}x^3$$

From the restrictions of $\sqrt{1-5x}$ and $(4x^2 + 1)$, we have

$$|x| < \frac{1}{5} \text{ and } |x| < \frac{1}{2}$$

The intersection is $|x| < \frac{1}{5}$.

e.g. Expand and give a simplified expression for $\sqrt[3]{9+3x}$ up to and including the term $x^3$. Write down the values of $x$ in which this expansion is valid.
\[ \sqrt{9 + 3x} = (9 + 3x)^{\frac{1}{3}} \]
\[ = 9^{\frac{1}{3}} \left(1 + \frac{1}{3}x\right)^{\frac{1}{3}} \]
\[ = 9^{\frac{1}{3}} \left(1 + \frac{1}{3} \left(\frac{1}{3} - \frac{2}{3} \frac{1}{2!} \left(\frac{1}{3} x\right)^2 + \frac{1}{3} \left(\frac{1}{3} \left(\frac{2}{3} \frac{1}{3} \left(\frac{5}{3} \left(\frac{1}{3} x\right)^3 + ... \right) \right) \right) \right) + \right) \]
\[ = 9^{\frac{1}{3}} \left(1 + \frac{1}{9} x - \frac{1}{81} x^2 + \frac{5}{2187} x^3 + ... \right) \]
\[ |x| < \frac{1}{\left(\frac{1}{3}\right)} \]
\[ |x| < 3 \]

Questions may further include:

- find an approximation by substituting a suitable value for \( x \) into the expansion
- approximate \( \int_{0}^{0.12} \frac{\sqrt{1 - 5x}}{4x^2 + 1} \, dx \)
- find \( p \), given the coefficient of a term (e.g. Given that the coefficient of the \( x^3 \) term in \((1 + px)^8\) is 448, find \( p \))
- simultaneous equations that involve coefficients
- discriminant, quadratic inequalities (very rare)
17 Trigonometric Equations

The CAST diagram above tells us which trigonometric ratio is positive in the four quadrants.

General solutions:

- \( \sin \theta = r \)
  \[
  \theta_{ref} = \sin^{-1} r \\
  \theta = \theta_{ref} + 2\pi k \text{ or } \pi - \theta_{ref} + 2\pi k, k = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (in radians)}
  \]
  \[
  \theta = \theta_{ref} + 360^\circ k \text{ or } 180^\circ - \theta_{ref} + 360^\circ k, k = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (in degrees)}
  \]

- \( \cos \theta = r \)
  \[
  \theta_{ref} = \cos^{-1} r \\
  \theta = \theta_{ref} + 2\pi k \text{ or } 2\pi - \theta_{ref} + 2\pi k, k = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (in radians)}
  \]
  \[
  \theta = \theta_{ref} + 360^\circ k \text{ or } 360^\circ - \theta_{ref} + 360^\circ k, k = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (in degrees)}
  \]

- \( \tan \theta = r \)
  \[
  \theta_{ref} = \tan^{-1} r \\
  \theta = \theta_{ref} + 2\pi k \text{ or } \pi + \theta_{ref} + 2\pi k, k = 0, \pm 1, \pm 2, \pm 3, \ldots
  \]
  \[
  \theta = \theta_{ref} + \pi k, k = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (in radians)}
  \]
  \[
  \theta = \theta_{ref} + 180^\circ k, k = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ (in degrees)}
  \]

Observe that:

- \( \theta_{ref} \) (reference angle) can be negative, it’s a just a particular solution
- \( \sin, \cos \) each has a period of \( 2\pi \), \( \tan \) has a period of \( \pi \)
- \( \tan \)’s solution can be combined into one single case, unlike \( \sin \) and \( \cos \)
- \( \pi = 180^\circ \) in angle measures
e.g. Solve $\sin 3x = -1$ for $-\pi \leq x \leq \pi$

$\sin 3x = -1$

$3x = \sin^{-1}(-1) + 2\pi k$ or $\pi - \sin^{-1}(-1) + 2\pi k, k = 0, \pm 1, \pm 2, \pm 3, \ldots$

$3x = \frac{-\pi}{2} + 2\pi k$ or $\pi - \frac{-\pi}{2} + 2\pi k, k = 0, \pm 1, \pm 2, \pm 3, \ldots$

$x = \frac{-\pi}{6} + \frac{2}{3}\pi k$ or $\frac{3\pi}{6} + \frac{2}{3}\pi k, k = 0, \pm 1, \pm 2, \pm 3, \ldots$

$x = \frac{-\pi}{6} + \frac{4}{6}\pi k$ or $\frac{3\pi}{6} + \frac{4}{6}\pi k, k = 0, \pm 1, \pm 2, \pm 3, \ldots$

$x = \frac{-5\pi}{6}, \frac{-\pi}{6}, \frac{\pi}{2} \left(= \frac{3\pi}{6}\right)$

By converting to common denominators, it’s easy to generate the desired solutions.

$-\pi \leq x \leq \pi \rightarrow -\frac{6\pi}{6} \leq x \leq \frac{6\pi}{6}$

To save time, I suggest writing down the following:

$\sin 3x = -1$

$3x = \frac{-\pi}{2}$ or $\pi - \frac{-\pi}{2} (\pm 2\pi k)$

$x = \frac{-\pi}{6}$ or $\frac{3\pi}{6} + \left(\pm \frac{2}{3}\pi k\right)$

$x = \frac{-\pi}{6}$ or $\frac{3\pi}{6} + \left(\pm \frac{4}{6}\pi k\right)$

$x = \frac{-5\pi}{6}, \frac{-\pi}{6}, \frac{\pi}{2}$

e.g. Solve $4\cos^2(\theta - 30^\circ) + 3\cos(\theta - 30^\circ) = 1$ for $-90^\circ \leq \theta \leq 180^\circ$, giving your answer correct to 1 decimal place.

$4\cos^2(\theta - 30^\circ) + 3\cos(\theta - 30^\circ) - 1 = 0$

$(4\cos(\theta - 30^\circ) - 1)(\cos(\theta - 30^\circ) + 1) = 0$

$4\cos(\theta - 30^\circ) - 1 = 0$ or $\cos(\theta - 30^\circ) + 1 = 0$

$\cos(\theta - 30^\circ) = \frac{1}{4}$ or $\cos(\theta - 30^\circ) = -1$

$\theta - 30^\circ = 75.522^\circ$ or $360^\circ - 75.522(\pm 360^\circ k)$

$\theta - 30^\circ = 180^\circ$ or $360^\circ - 180^\circ (\pm 360^\circ k)$

$\theta = 105.522, 314.478, 210(\pm 360^\circ k)$

$\theta = -45.5^\circ, 105.5^\circ$
e.g. Solve \( \sin^2 \theta = 2 \cos^2 \theta \), for \( \frac{-3\pi}{2} \leq \theta \leq \pi \), giving your answers correct to 3 significant figures.

\[
\sin^2 \theta = 2 \cos^2 \theta
\]

\[
\frac{\sin^2 \theta}{\cos^2 \theta} = 2
\]

\[
\tan^2 \theta = 2
\]

\[
\tan \theta = \pm \sqrt{2}
\]

\[
\tan \theta = \sqrt{2} \quad \text{or} \quad \tan \theta = -\sqrt{2}
\]

\[
\theta = 0.95532(\pm \pi k)
\]

\[
\theta = -0.95532(\pm \pi k)
\]

\[
\theta = -4.10, -2.19, -0.955, 0.955, 2.19
\]

You should substitute all values back into the calculator to see if they are correct. Plot the function on a GDC by setting the original equation to 0 and check how many roots are there to ensure you have the correct number of answers.
18 Trigonometric Identities

You need to know:

- \( \sin^2 x + \cos^2 x = 1 \)
- \( \tan x = \frac{\sin x}{\cos x} \)

The following will be given when needed, but it's good to know the results and how to prove:

- \( \sin(A + B) = \sin A \cos B + \cos A \sin B \)
- \( \sin(A - B) = \sin A \cos B - \cos A \sin B \)
- \( \cos(A + B) = \cos A \cos B - \sin A \sin B \)
- \( \cos(A - B) = \cos A \cos B + \sin A \sin B \)
- \( \tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} \)
  \[= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \]
  \[= \frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B} \]
  \[= \cos A + \tan A \tan B \]

Know this proof. Key point is remembering to divide by \( \cos A \cos B \). The hint is in the bottom left term in the denominator because it needs to go to 1 in the end.

- \( \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \)

Similar proof as above, just the first step with \( \frac{\sin(A - B)}{\cos(A - B)} \).
• \( \sin 2A = \sin(A + A) \)
  \[= \sin A \cos A + \cos A \sin A \]
  \[= 2 \sin A \cos A \]

• \( \cos 2A = \cos(A + A) \)
  \[= \cos A \cos A - \sin A \sin A \]
  \[= \cos^2 A - \sin^2 A \]
  \[= (1 - \sin^2 A) - \sin^2 A \]
  \[= 1 - 2 \sin^2 A \]
  \[= 2 \cos^2 A - 1 \]

• \( \tan 2A = \tan(A + A) \)
  \[= \frac{\tan A + \tan A}{1 - \tan A \tan A} \]
  \[= \frac{2 \tan A}{1 - \tan^2 A} \]

• \( \cos 2A = 1 - 2 \sin^2 A \)
  \[\text{or} \quad \cos 2A = 2 \cos^2 A - 1 \]

These results are important when we want to integrate \( \sin^2 x \) or \( \cos^2 x \).

\[ \text{i.e.} \; \int \sin^2 x \, dx = \int \frac{1}{2} (1 - \cos 2A) \, dx = \frac{1}{2} x - \frac{1}{4} \sin 2A + C \]

\[ \int \cos^2 x \, dx = \int \frac{1}{2} (1 + \cos 2A) \, dx = \frac{1}{2} x + \frac{1}{4} \sin 2A + C \]

In exams, the last line is usually asked to be shown to be true, so:

• \( \sin^2 A = \frac{1}{2} (1 - \cos 2A) \)
  \[\text{similarly} \quad \cos^2 A = \frac{1}{2} (1 + \cos 2A) \]
  \[\quad = \frac{1}{2} \left( 1 - (\cos^2 A - \sin^2 A) \right) \]
  \[\quad = \frac{1}{2} \left( 1 - ((1 - \sin^2 A) - \sin^2 A) \right) \]
  \[\quad = \frac{1}{2} \left( 1 - (1 - 2 \sin^2 A) \right) \]
  \[\quad = \frac{1}{2} (2 \sin^2 A) \]
  \[\quad = \sin^2 A \]

LHS = RHS

\[ \text{LHS} = \text{RHS} \]
• \( \sin 3A = \sin(2A + A) \)
  \[
  = \sin 2A \cos A + \cos 2A \sin A \\
  = (2 \sin A \cos A) \cos A + (1 - 2 \sin^2 A) \sin A \\
  = \sin A(2 \cos^2 A + (1 - 2 \sin^2 A)) \\
  = \sin A(2(1 - \sin^2 A) + 1 - 2 \sin^2 A) \\
  = \sin A(3 - 4 \sin^2 A) \\
  = 3 \sin A - 4 \sin^3 A
  \]

• \( \cos 3A = \cos(2A + A) \)
  \[
  = \cos 2A \cos A - \sin 2A \sin A \\
  = (2 \cos^2 A - 1) \cos A - (2 \sin A \cos A)(\sin A) \\
  = \cos A(2 \cos^2 A - 1 - 2 \sin^2 A) \\
  = \cos A(2 \cos^2 A - 1 - 2(1 - \cos^2 A)) \\
  = \cos A(4 \cos^2 A - 3) \\
  = 4 \cos^3 A - 3 \cos A
  \]

• \( \tan 3A = \tan(2A + A) \)
  \[
  = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} \\
  = \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A}{1 - \tan^2 A} \tan A} \\
  = \frac{2 \tan A - \tan A(1 - \tan^2 A)}{1 - \tan^2 A + \frac{2 \tan^2 A}{1 - \tan^2 A}} \\
  = \frac{3 \tan A - \tan^2 A}{1 - 3 \tan^3 A}
  \]
\[
\begin{align*}
\sin^2 x + \cos^2 x &= 1 \\
\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
\tan^2 x + 1 &= \frac{1}{\cos^2 x}
\end{align*}
\]

Reversely,

\[
\begin{align*}
\tan^2 x + 1 &= \frac{1}{\cos^2 x} \\
\frac{\sin^2 x}{\cos^2 x} + 1 &= \frac{1}{\cos^2 x} \\
\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
\frac{\sin^2 x + \cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
\frac{1}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
\text{LHS} &= \text{RHS}
\end{align*}
\]
19 Trigonometry

• sine rule: \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \);
  \( \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \)

• cosine rule: \( c^2 = a^2 + b^2 - 2ab \cos C \);
  \( C = \cos^{-1} \left( \frac{a^2 + b^2 - c^2}{2ab} \right) \)

• area of triangle using sine: \( \frac{1}{2}ab \sin C \)

19.1 Ambiguous Triangle

When using sine rule, there are two potential answers when solving for an unknown angle. This does NOT apply for unknown side as you would be given 2 angles and a side, hence the 3rd angle is fixed.

The second solution comes straight from solving sine equations:

\( \theta \) or \( 180^\circ - \theta \)

19.2 3D Trigonometry

Draw in lines and angles to help you better visualise what you need to solve. In a lot of cases, you need to construct the diagonals of the base.
To find the area underneath a curve, we evaluate the following:

\[
\int_a^b f(x) \, dx = \left[ F(x) \right]_a^b = F(b) - F(a)
\]

Special care needs to be taken if the area falls under the x-axis. In which case, we need to subtract the definite integral to ensure a positive value representing the area.

\[
\text{area} = \int_a^b y \, dx - \int_b^c y \, dx = \int_a^c |y| \, dx
\]

You can plug this statement in the calculator to check the final answer.
e.g. Find the area bounded by the curve \( y = 3 + 5x - 2x^2 \) and the \( x \)-axis.

Do a sketch of the curve.

\[
\begin{array}{c}
\text{Shade in the area of interest.}
\end{array}
\]

Notice that we need the \( x \)-coordinates of the intersections with the \( x \)-axis.

\[
\begin{align*}
3 + 5x - 2x^2 &= 0 \\
-2x^2 + 5x + 3 &= 0 \\
-(2x^2 - 5x - 3) &= 0 \\
-(2x + 1)(x - 3) &= 0 \\
x &= -\frac{1}{2}, 3
\end{align*}
\]

Integrate

\[
\int_{\frac{-1}{2}}^{3} (3 + 5x - 2x^2) \, dx = \left[ 3x + \frac{5x^2}{2} - \frac{2x^3}{3} \right]_{\frac{-1}{2}}^{3}
\]

\[
= \left( 3(3) + \frac{5(3)^2}{2} - \frac{2(3)^3}{3} \right) - \left( 3 \left( \frac{-1}{2} \right) + \frac{5 \left( \frac{-1}{2} \right)^2}{2} - \frac{2 \left( \frac{-1}{2} \right)^3}{3} \right)
\]

\[
= \frac{27}{2} - \left( \frac{-19}{24} \right)
\]

\[
= \frac{343}{24}
\]
e.g. Given that \((x - 3)\) is a factor in \(y = x^3 - 4x^2 + x + 6\).

Find the area bounded by the curve \(y\) and the \(x\)-axis.

\[
x^2 - x - 2
\]
\[
(x - 3)(x^3 - 4x^2 + x + 6)
\]
\[
x^3 - 3x^2
\]
\[
-x^2 + x
\]
\[
-x^2 + 3x
\]
\[
-2x + 6
\]
\[
-2x + 6
\]
\[
0
\]
\[
y = (x - 3)(x^2 - x - 2)
\]
\[
= (x - 3)(x - 2)(x + 1)
\]

area = \(\int_a^b y \, dx - \int_b^c y \, dx\)

\[
= \int_{-1}^{2} (x^3 - 4x^2 + x + 6) \, dx - \int_{2}^{3} (x^3 - 4x^2 + x + 6) \, dx
\]

\[
= \left[ \frac{x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} + 6x \right]_{-1}^{2} - \left[ \frac{x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} + 6x \right]_{2}^{3}
\]

\[
= \left( \frac{(2)^3}{4} - \frac{4(2)^3}{3} + \frac{(2)^2}{2} + 6(2) \right) - \left( \frac{(-1)^3}{4} - \frac{4(-1)^3}{3} + \frac{(-1)^2}{2} + 6(-1) \right)
\]

\[
- \left( \frac{(3)^3}{4} - \frac{4(3)^3}{3} + \frac{(3)^2}{2} + 6(3) \right) - \left( \frac{(2)^3}{4} - \frac{4(2)^3}{3} + \frac{(2)^2}{2} + 6(2) \right)
\]

\[
= \left( \frac{22}{3} \right) - \left( -\frac{47}{12} \right) - \left( \frac{27}{4} \right) - \left( \frac{22}{3} \right)
\]

\[
= \frac{45}{4} - \frac{7}{12}
\]

\[
= 71 \frac{1}{6}
\]

Check that:

\[
\int_{-1}^{3} |x^3 - 4x^2 + x + 6| \, dx = 71 \frac{1}{6} \checkmark
\]
21 Area between Curves

The area between two curves can be obtained by:

\[
\text{area below the top curve} - \text{area below the bottom curve} = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b [f(x) - g(x)] \, dx = \int_a^b (y_1 - y_2) \, dx
\]

In a way, you can think of area below a curve, bounded by the \(x\)-axis, as \(y_1 = f(x)\) being the top curve, and \(y_2 = 0\) being the bottom:

\[
\int_a^b [f(x) - g(x)] \, dx = \int_a^b [f(x) - 0] \, dx = \int_a^b f(x) \, dx
\]
This is why if the area is below the $x$-axis, we subtract the integral to obtain a positive area.

\[
\int_a^b [f(x) - g(x)] \, dx \\
\int_a^b [0 - g(x)] \, dx \\
= \int_a^b -g(x) \, dx = - \int_a^b g(x) \, dx
\]
Figure 2 shows the curve $y = x^3 - 5x^2 - 2x + 24$. The line $l$ is the tangent to the curve at $x = \frac{1}{2}$.

(a) Show that the line $l$ crosses the $x$-axis at $x = 4$.

(b) Find the area bounded by the curve $y$ and the line $l$.

Solution:

(a) $\frac{dy}{dx} = 3x^2 - 10x - 2$

$\left. \frac{dy}{dx} \right|_{x=\frac{1}{2}} = 3 \left( \frac{1}{2} \right)^2 - 10 \left( \frac{1}{2} \right) - 2$

$= \frac{-25}{4}$

When $x = \frac{1}{2}$, $y = \left( \frac{1}{2} \right)^3 - 5 \left( \frac{1}{2} \right)^2 - 2 \left( \frac{1}{2} \right) + 24$

$= \frac{175}{8}$

$l$ passes through $\left( \frac{1}{2}, \frac{175}{8} \right)$. 
Equation of $l$ is:
\[
y - \frac{175}{8} = \frac{-25}{4} \left( x - \frac{1}{2} \right)
\]
\[
8y - 175 = -50 \left( x - \frac{1}{2} \right)
\]
\[
8y - 175 = -50x + 25
\]
\[
8y = -50x + 200
\]
\[
y = \frac{-25}{4} x + 25
\]
When $x = 4$, $y = \frac{-25}{4} (4) + 25$
\[
= 0
\]
$l$ crosses the $x$-axis at $x = 4$.

(b) Area:
\[
\text{area} = \int_{\frac{1}{2}}^{4} \left[ \left( \frac{-25}{4} x + 25 \right) - (x^3 - 5x^2 - 2x + 24) \right] dx
\]
\[
= \int_{\frac{1}{2}}^{4} \left( -x^3 + 5x^2 - \frac{17}{4} x + 1 \right) dx
\]
\[
= \left[ -\frac{x^4}{4} + \frac{5x^3}{3} - \frac{17x^2}{8} + x \right]_{\frac{1}{2}}^{4}
\]
\[
= \left[ -\frac{(4)^4}{4} + \frac{5(4)^3}{3} - \frac{17(4)^2}{8} + (4) \right] - \left[ -\frac{\left( \frac{1}{2} \right)^4}{4} + \frac{5\left( \frac{1}{2} \right)^3}{3} - \frac{17\left( \frac{1}{2} \right)^2}{8} + \left( \frac{1}{2} \right) \right]
\]
\[
= \frac{38}{3} - \frac{31}{192}
\]
\[
= \frac{2401}{192}
\]
22 Volume by Revolution

Suppose we want to obtain a solid shape by revolving a curve about the $x$-axis by $360^\circ$. We can add up all the disks (in the form of cylinders) via the integral:

\[
\int_a^b \pi [f(x)]^2 \, dx = \int_a^b \pi y^2 \, dx = \pi \int_a^b y^2 \, dx
\]

If the region to be revolved is bounded between two curves, say $y_1$ and $y_2$, then

\[
V = \int_a^b \pi (y_2)^2 \, dy - \int_a^b \pi (y_1)^2 \, dy = \int_a^b \pi (y_2^2 - y_1^2) \, dy = \pi \int_a^b (y_2^2 - y_1^2) \, dy
\]

where $y_2$ lies above $y_1$ in the interval of interest.

Use basic shapes for integration whenever possible, in particular, horizontal or slanted lines.
Let’s give some values to the previous example:

\[ y = \sqrt{x} \]

Suppose we are interested in finding the volume obtained by revolving the curve \( y = \sqrt{x} \) upon the \( x \)-axis by 360° and between the lines \( x = 1 \) and \( x = 3 \). Then we have:

\[
V = \int_1^3 \pi (\sqrt{x})^2 \, dx
\]

\[
= \pi \int_1^3 x \, dx
\]

\[
= \pi \left[ \frac{x^2}{2} \right]_1^3
\]

\[
= \pi \left[ \left( \frac{(3)^2}{2} \right) - \left( \frac{(1)^2}{2} \right) \right]
\]

\[
= \pi \left( \frac{9}{2} - \frac{1}{2} \right)
\]

\[
= \pi (4)
\]

\[
= 4\pi
\]
Figure 3 shows part of the curve with equation \( y = 2\sin x \). \( P \) is the point with coordinates \((\frac{\pi}{3}, \sqrt{3})\).

The normal to the curve at \( P \) cuts the \( x \)-axis at \( A \).

(a) Show that an equation of the normal \( AP \) is

\[
y + x = \frac{\pi}{3} + \sqrt{3}.
\]

(b) Show that \( \frac{d}{dx}(2x - \sin 2x) = 4\sin^2 x \).

The shaded area is rotated through 360° about the \( x \)-axis.

(c) Using the result of (b), or otherwise, calculate, in terms of \( \pi \), the volume of the solid generated.

Try it yourself and then read the solution:
Solution:

(a) 

\[
\frac{dy}{dx} = 2 \cos x \\
\left. \frac{dy}{dx} \right|_{x=\frac{\pi}{3}} = 2 \cos \frac{\pi}{3} = 1
\]

gradient of normal \[\frac{-1}{\left. \frac{dy}{dx} \right|_{x=\frac{\pi}{3}}}} = \frac{-1}{1} = -1
\]

Equation of normal \( AP \):

\[
y - \sqrt{3} = -1 (x - \frac{\pi}{3}) \\
y - \sqrt{3} = -x + \frac{\pi}{3} \\
\]

\[
y + x = \frac{\pi}{3} + \sqrt{3}
\]

(b) 

\[
\frac{d}{dx} (2x - \sin 2x) = 2 - 2 \cos 2x \\
= 2 - 2(1 - 2 \sin^2 x) \\
= 2 - 2 + 4 \sin^2 x \\
= 4 \sin^2 x
\]

(c) 

When \( y = 0 \), from the normal \( AP \),

\[
0 + x = \frac{\pi}{3} + \sqrt{3} \\
x = \frac{\pi}{3} + \sqrt{3}
\]

\( A \) has \( x \)-coordinate \( \left( \frac{\pi}{3} + \sqrt{3} \right) \)
Before you freak out, of course there’s an easier way. But hey, at least the standard procedure works! But please use basic geometry shapes whenever possible.
Let’s break down the two integrals:

Volume = \[ \int_0^\pi \pi (2 \sin x)^2 \, dx + \int_{\pi/3}^{\pi+\sqrt{3}} \pi \left( \frac{\pi}{3} + \sqrt{3} - x \right)^2 \, dx \]

\[ \int_0^\pi \pi (2 \sin x)^2 \, dx = \pi \int_0^\pi 4 \sin^2 x \, dx \]

\[ = \pi \left[ 2x - \sin 2x \right]_0^\pi \]

\[ = \pi \left[ 2 \left( \frac{\pi}{3} \right) - \sin 2 \left( \frac{\pi}{3} \right) \right] - \left[ 2(0) - \sin 2(0) \right] \]

\[ = \pi \left[ \left( \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) - (0) \right] \]

\[ = \left[ \left( \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \right] \pi \]
The region that \( \int_{\pi/3}^{\pi+\sqrt{3}} \pi \left( \frac{\pi}{3} + \sqrt{3} - x \right)^2 \, dx \) captures is actually just a cone pointing to the right with height \( \left( \frac{\pi}{3} + \sqrt{3} - \frac{\pi}{3} = \sqrt{3} \right) \), radius \( \sqrt{3} \).

Hence,

\[
\int_{\pi/3}^{\pi+\sqrt{3}} \pi \left( \frac{\pi}{3} + \sqrt{3} - x \right)^2 \, dx = \frac{\pi(\sqrt{3})^2(\sqrt{3})}{3} = \frac{3\sqrt{3}\pi}{3} = \sqrt{3}\pi \quad \text{This looks familiar, doesn’t it?}
\]
Combining the two volumes:

\[
\left[ \left( \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \right] \pi + \sqrt{3}\pi \\
= \left[ \left( \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) + \sqrt{3} \right] \pi \\
= \left[ \frac{2\pi}{3} + \frac{\sqrt{3}}{2} \right] \pi
\]

If you got it right, congratulations! Here’s a chocolate-dipped strawberry ice-cream cone, or chocolate cupcake, whichever you fancy.
22.1 $y$-axis

Similar procedure follows for revolution upon the $y$-axis.

\[
V = \int_a^b \pi [f^{-1}(y)]^2 \, dy \\
= \int_a^b \pi x^2 \, dy \\
= \pi \int_a^b x^2 \, dy
\]

Questions are usually given in the form where $x^2$ can be isolated easily. An immediate step may involve solving for the lower and upper limits, $a$ and $b$, respectively, before integrating it.

If the region lies between curves, say $y_1$ and $y_2$, you may need to subtract the volumes to obtain the final answer. Namely,

\[
\int_a^b \pi (x_2)^2 \, dy - \int_a^b \pi (x_1)^2 \, dy \\
= \int_a^b \pi (x_2^2 - x_1^2) \, dy \\
= \pi \int_a^b (x_2^2 - x_1^2) \, dy
\]

where $y_2$ lies above $y_1$ in the interval of interest.
23 Kinematics

All of the above are assumed to be functions in terms of time \( t \). (i.e. \( s = s(t), v = v(t), a = a(t) \))

We sometimes use subscripts as shorthand. i.e. \( (s_0 = s(0), v(2) = v_2, s_t = s(t) = s, \text{etc.})\)

Kinematics Formulae:

\[
\begin{align*}
  v &= \frac{ds}{dt} \\
  v(t_i) &= \left. \frac{ds}{dt} \right|_{t=t_i} \\
  s &= \int v \, dt \\
  d &= \int |v| \, dt \\
  \text{speed} &= |v| \\
  a &= \frac{dv}{dt} \\
  a(t_i) &= \left. \frac{dv}{dt} \right|_{t=t_i} \\
  \Delta s &= \int_{t_1}^{t_2} v \, dt \\
  &= s(t_2) - s(t_1)
\end{align*}
\]

Note: \( s \) is the spatium (Latin for space) of an object which describes its position.
In this course, we simplify things a bit.

When dealing with displacement, it’s assumed to be relative to \( s_0 \), which implies \( t_1 = t_0 = 0 \), effectively rendering position and displacement to be the same.

Distance will be between two time points, because finding the antiderivative for the general case is beyond the scope of this course. To my knowledge, it is rarely done in practice.

![Kinematics Flowchart](image)

Further Pure Mathematics Kinematics Flowchart

**Kinematics Formulae:**

\[
\begin{align*}
v &= \frac{ds}{dt} \\
v(t_i) &= \frac{ds}{dt} \bigg|_{t=t_i} \\
S &= \int v \, dt \\
S &= \int_{t_1}^{t_2} |v| \, dt \\
a &= \frac{dv}{dt} \\
&= \frac{d^2s}{dt^2} \\
a(t_i) &= \frac{dv}{dt} \bigg|_{t=t_i} \\
a(t_i) &= \frac{d^2s}{dt^2} \bigg|_{t=t_i} \\
d &= \int_{t_1}^{t_2} |v| \, dt
\end{align*}
\]

Finding the maximum or minimum velocity requires setting \( \frac{ds}{dt} = 0 \) or \( v = 0 \) to find critical values and **checking the endpoints** \( (t_1 \text{ and } t_2) - \text{Absolute Extrema} \).

Finding the maximum speed requires taking the absolute value (magnitude) of the velocity values above and pick out the greatest. Minimum requires taking the least, but the lowest being 0.
e.g. The velocity of a particle, $v$ m/s, is given by the equation

$$v = 3t^2 - t - 2, t \geq 0$$

(a) Find the time when the particle is instantaneously at rest.

(b) Find the acceleration of the particle when $t = 2$.

(c) Find the distance travelled until the particle stops momentarily.

Solution:

(a) Set $v = 0$,

$$3t^2 - t - 2 = 0$$

$$(3t + 2)(t - 1) = 0$$

$$t = \frac{-2}{3} \text{ (reject) or } t = 1$$

$$t = 1$$

(b)

$$\frac{dv}{dt} = 6t - 1$$

$$\frac{dv}{dt} \bigg|_{t=2} = 6(2) - 1$$

$$= 11 \text{ m/s}^2$$

(c) Sketch the velocity curve.

Notice that the area that we want is below the $x$-axis. Therefore,

$$d = \int_0^1 |3t^2 - t - 2| \, dt$$

$$= -\int_0^1 (3t^2 - t - 2) \, dt \quad \text{below the } x\text{-axis}$$

$$= - \left[ t^3 - \frac{t^2}{2} - 2t \right]_0^1$$

$$= - \left[ \left( (1)^3 - \frac{(1)^2}{2} - 2(1) \right) - 0 \right]$$

$$= - \left( \frac{-3}{2} \right)$$

$$= \frac{3}{2} \text{ m}$$
e.g. The displacement of a particle, $s$ m, is given by the equation

$$ s = \sin^2 x + \cos x $$

Find the acceleration of the particle when $t = \frac{\pi}{2}$.

Solution:

$$ \frac{ds}{dt} = 2 \sin x \cdot \cos x - \sin x $$

$$ = \sin 2x - \sin x $$

$$ \frac{d^2s}{dt^2} = 2 \cos 2x - \cos x $$

$$ \left. \frac{d^2s}{dt^2} \right|_{t=\frac{\pi}{2}} = 2 \cos \left( \frac{\pi}{2} \right) - \cos \left( \frac{\pi}{2} \right) $$

$$ = 2(-1) - (0) $$

$$ = -2 \text{ m/s}^2 $$

e.g. A gamma particle moves along the $x$-axis so that at time $t$ seconds its displacement from $O$ is $x$ metres. Its velocity is given by the equation

$$ v = e^{-0.5t} + 2, \quad t \geq 0 $$

The displacement of the particle at $t = 0$ is 4 m. Find an expression of $x$ in terms of $t$.

Solution:

$$ x = \int e^{-0.5t} + 2 \, dt $$

$$ = \frac{e^{-0.5t}}{-0.5} + 2t + C $$

$$ = -2e^{-0.5t} + 2t + C $$

When $t = 0$, $x = 4$,

$$ 4 = -2e^{-0.5(0)} + 2(0) + C $$

$$ 4 = -2(1) + 0 + C $$

$$ 6 = C $$

$$ x = -2e^{-0.5t} + 2t + 6 $$
The velocity of a car, \( v \) m/s, is given by the equation

\[
v = 5x - 2x^2, \quad t \geq 0
\]

(a) Find the maximum speed during the first four seconds.

(b) Find the distance travelled during the first four seconds.

Solution:

(a)

\[
\frac{dv}{dt} = 5 - 4x
\]

Set \( \frac{dv}{dt} = 0 \)

\[
5 - 4x = 0
\]

\[
5 = 4x
\]

\[
\frac{5}{4} = x
\]

\[
v\left(\frac{5}{4}\right) = 5\left(\frac{5}{4}\right) - 2\left(\frac{5}{4}\right)^2 = \frac{25}{8} \text{ m/s}
\]

\[
v(0) = 5(0) - 2(0)^2 = 0 \text{ m/s}
\]

\[
v(4) = 5(4) - 2(4)^2 = -12 \text{ m/s}
\]

The maximum speed is 12 m/s.

Note: Although not asked, the maximum velocity is \( \frac{25}{8} \) m/s.
(b) Sketch the velocity curve.

\[ 5x - 2x^2 = 0 \]
\[ x(5 - 2x) = 0 \]
\[ x = 0, \frac{5}{2} \]

\[
d = \int_{0}^{4} |5x - 2x^2| \, dt
\]
\[
= \int_{0}^{\frac{5}{2}} (5x - 2x^2) \, dt - \int_{\frac{5}{2}}^{4} (5x - 2x^2) \, dt
\]
\[
= \left[ \frac{5x^2}{2} - \frac{2x^3}{3} \right]_{0}^{\frac{5}{2}} - \left[ \frac{5x^2}{2} - \frac{2x^3}{3} \right]_{\frac{5}{2}}^{4}
\]
\[
= \left[ \left( \frac{5 \left( \frac{5}{2} \right)^2}{2} - \frac{2 \left( \frac{5}{2} \right)^3}{3} \right) - (0) \right] - \left[ \left( \frac{5(4)^2}{2} - \frac{2(4)^3}{3} \right) - \left( \frac{5 \left( \frac{5}{2} \right)^2}{2} - \frac{2 \left( \frac{5}{2} \right)^3}{3} \right) \right]
\]
\[
= \frac{125}{24} - \left( \frac{-8}{3} - \frac{125}{24} \right)
\]
\[
= \frac{157}{12} \text{ m}
\]
## Exponential and Logarithm

### Table for Exponentials and Logarithms

\[
\begin{align*}
    x^m \cdot x^n &= x^{m+n} & \log a + \log b &= \log ab \\
    \frac{x^m}{x^n} &= x^{m-n} & \log a - \log b &= \log \frac{a}{b} \\
    x^0 &= 1, \quad x \neq 0 & \log 1 &= 0 \\
    (x^m)^n &= x^{mn} & \log a^n &= n \log a \\
    (xy)^n &= x^n \cdot y^n & \log_{xy} a &= \frac{\log a}{\log x} \\
    \left(\frac{x}{y}\right)^n &= \left(\frac{x^n}{y^n}\right) & \left(\frac{x}{y}\right)^n &= \left(\frac{y}{x}\right)^n \\
    \frac{1}{x^n} &= x^{-n} & \frac{1}{x^n} &= x^{-n} \\
    \frac{1}{x^{-n}} &= x^n & \frac{1}{x^{-n}} &= x^n \\
    \left(\frac{x}{y}\right)^{-n} &= \left(\frac{y}{x}\right)^n & \log_{x^n} a &= \log_x \sqrt[n]{a} \\
    x^n &= \sqrt[n]{x^m} & \log_{x^n} a &= \log_x \sqrt[n]{a} \\
    \log 1 &= 0, \log 10 = 1, \log 100 = 2, \log 1000 = 3, ... & \log 2 = 0, \log 2 \cdot 2 = 1, \log 2 \cdot 4 = 2, \log 2 \cdot 8 = 3, ... \\
    \ln 1 &= 0, \ln e = 1, \ln e^2 = 2, \ln e^3 = 3, ... \\
    \ln a^n &= n \ln a \\
    \ln x a &= \frac{\ln a}{\ln x} \\
    \ln x n &= \ln x \sqrt[n]{a} \\
    \ln \sqrt[n]{a} &= \frac{1}{n} \ln a \\
    \ln e &= 1 \\
    \ln e^2 &= 2 \\
    \ln e^3 &= 3 \\
\end{align*} \]

### 24.1 \(e\) as a mathematical constant

\(e\) is a constant that was first defined for the effective interest rate of continuously compounded interest.

\[
e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \\
\approx 2.718281828
\]

Using base \(e\) for the exponential model turns out to have some nice properties, namely easy differentiation and integration. Many models use \(e\) and transformations are applied afterwards to give us better fit, such as:

- **growth and decay:** \(P(t) = Ae^{kt} + P_0\) or \(P(t) = Ae^{-kt} + P_0\) (population)
- **continuous compound interest:** \(P(t) = P_0e^{rt}\) (principal)
- **standard normal distribution:** \(\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\) (probability)
24.2 Cancellation Properties

\[ a^x = b \iff \log_a b = x \]

This is the usual definition of logarithm but its use is quite limited. For example:

\[ \log_2 256 = 8 \text{ since } 2^8 = 256. \]

But \( \log_2 5 = ? \text{ since } 2^? = 5 \). Logarithm can then help us look for solutions in this form.

From the definition, if we substitute \( x = \log_a b \), we get:

\[ a^{\log_a b} = b \]

Similarly, if we substitute \( b = a^x \), we get:

\[ \log_a a^x = x \]

From the above cancellation properties, we can deduce that exponential and logarithm are inverse functions! Recall from last year that:

\[ f^{-1}(f(x)) = x \text{ and } f^{-1}(x) = f(x) \]

Since exponential and logarithm are both monotonic (strictly increasing or decreasing), we can take the logarithm of or raise each side of the equation or inequality without worrying the change of sign.

e.g. Solve \( 2^{x-3} = 7 \).

Solution:

\[ 2^{x-3} = 7 \]

\[ \log 2^{x-3} = \log 7 \]

\[ (x - 3) \log 2 = \log 7 \]

\[ x - 3 = \frac{\log 7}{\log 2} \]

\[ x = \frac{\log 7}{\log 2} + 3 \]

\[ \approx 5.807354922 \,(5.81) \]
e.g. Solve \( 9^x \times \frac{3^x}{\sqrt{3}} - 1 = 0 \)

Solution:
\[
9^x \times \frac{3^x}{\sqrt{3}} - 1 = 0 \\
3^{2x} \times \frac{3^x}{3^\frac{1}{2}} = 1 \\
3^{2x} \times 3^{x-\frac{1}{2}} = 3^0 \\
3^{2x+(x-\frac{1}{2})} = 3^0 \\
3^{3x-\frac{1}{2}} = 3^0 \\
3x - \frac{1}{2} = 0 \\
x = \frac{1}{6}
\]

e.g. [Disguised Quadratic] Solve \( 3^{2x+1} - 10(3^x) + 8 = 0 \)

Solution:
\[
3^{2x+1} - 10(3^x) + 8 = 0 \\
3^{2x} \cdot 3 - 10(3^x) + 8 = 0 \\
3 \cdot 3^{2x} - 10(3^x) + 8 = 0
\]

Let \( y = 3^x \), then
\[
3y^2 - 10y + 8 = 0 \\
(3y - 4)(y - 2) = 0 \\
y = \frac{4}{3} \quad \text{or} \quad y = 2 \\
3^x = \frac{4}{3} \quad \text{or} \quad 3^x = 2 \\
x = \frac{\log 4}{\log 3} \quad \text{or} \quad x = \frac{\log 2}{\log 3} \\
x \approx 0.631 \quad \text{or} \quad x \approx 0.262
e.g. Given that $\log_3 p = x, \log_3 q = y$, express $\log_9 \left( \frac{p^3 \sqrt{q}}{27} \right)$ in terms of $x$ and $y$.

Solution:

\[
\log_9 \left( \frac{p^3 \sqrt{q}}{27} \right) = \frac{\log \left( \frac{p^3 \sqrt{q}}{27} \right)}{\log 9} = \frac{\log \left( \frac{p^3 \sqrt{q}}{27} \right)}{2 \log 3} = \frac{1}{2} \left( \log_3 \left( \frac{p^3 \sqrt{q}}{27} \right) \right) = \frac{1}{2} \left( \log_3 p^3 + \log_3 \sqrt{q} - \log_3 27 \right) = \frac{1}{2} \left( \log_3 p^3 + \log_3 q^{\frac{1}{2}} - \log_3 3^3 \right) = \frac{1}{2} \left( 3 \log_3 p + \frac{1}{2} \log_3 q - 3 \log_3 3 \right) = \frac{1}{2} \left( 3x + \frac{1}{2}y - 3 \right) = \frac{3}{2}x + \frac{1}{4}y - \frac{3}{2}
\]

e.g. [Disguised Quadratic] Solve $\log_x 9 + 6 \log_9 x = 5$

Solution:

\[
\log_x 9 + 6 \log_9 x = 5
\]

Let $y = \log_9 x$, then

\[
\frac{1}{y} + 6y = 5
\]

\[
1 + 6y^2 = 5y
\]

\[
6y^2 - 5y + 1 = 0
\]

\[
(3y - 1)(2y - 1) = 0
\]

\[
y = \frac{1}{3} \quad \text{or} \quad y = \frac{1}{2}
\]

\[
\log_9 x = \frac{1}{3} \quad \log_9 x = \frac{1}{2}
\]

\[
x = 9^{\frac{1}{3}} \quad x = 9^{\frac{1}{2}}
\]

\[
= \sqrt[3]{9} \quad = 3
\]

\[
\approx 2.08
\]
25 Cubic Functions

A cubic equation

\[ y = ax^3 + bx^2 + cx + d \quad a, b, c, d \in \mathbb{R} \]

can always be written as

\[ y = a(x - \alpha_1)(x^2 + ex + f) \quad a, e, f, \alpha_1 \in \mathbb{R} \]

and sometimes

\[ y = a(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \quad a, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \]

Several techniques help us deduce whether a number \( p \) is a root or not, how to reduce a polynomial if such a root \( p \) is found, and how to select such candidates of \( p \).

25.1 Remainder Theorem

When dividing a polynomial \( p(x) \) by a linear factor \((ax - b)\), it can be expressed in the following quotient-remainder form:

\[ p(x) = (ax - b)q(x) + r(x) \]

Think about how you can write \( 13 \div 4 \) as \( 13 = 3 \times 4 + 1 \).

To find the remainder, we substitute \( x = \frac{b}{a} \), observe that

\[
p\left(\frac{b}{a}\right) = \left(a \left(\frac{b}{a}\right) - b\right)q\left(\frac{b}{a}\right) + r \left(\frac{b}{a}\right)
= r \left(\frac{b}{a}\right)
\]

When \( a = 1 \), the linear factor will often be in the form \((x - b)\). So you will see \( p(b) \) used more often than \( p\left(\frac{b}{a}\right) \).

e.g. Find the remainder of \( f(x) = x^3 - x + 2 \) when divided by \((x + 2)\).

Solution:

\[ f(-2) = (-2)^3 - (-2) + 2 = -4. \]

The remainder is \(-4\).
e.g. Show that \((2x - 1)\) is a factor in \(y = 8x^3 - 14x^2 - 7x + 6\).

Solution:
\[
y = f\left(\frac{1}{2}\right) = 8\left(\frac{1}{2}\right)^3 - 14\left(\frac{1}{2}\right)^2 - 7\left(\frac{1}{2}\right) + 6
\]
\[
= 1 - \frac{7}{2} - \frac{7}{2} + 6
\]
\[
= 0
\]
e.g. Show that \(x = -3\) is a root in \(f(x) = (x^2 + 5x + 6)(x - 3)\)

Solution:
\[
f(-3) = \left((-3)^2 + 5(-3) + 6\right)[(-3) - 3]
\]
\[
= (0)(-6)
\]
\[
= 0
\]
You should notice that \(x^2 + 5x + 6 = (x + 2)(x + 3)\), which also shows \(x = -3\) is a root.

### 25.2 Polynomial Division

We can employ polynomial division to reduce a polynomial once a root is found. This works similarly to long division with a few caveats.

e.g. Given that \((x - 3)\) is a factor in \(y = x^3 - 3x^2 - 4x + 12\), factorise completely.

Solution:
\[
x^2 - 4
\]
\[
x - 3 \overrightarrow{x^3 - 3x^2 - 4x + 12}
\]
\[
x^3 - 3x^2
\]
\[
4x + 12
\]
\[
0
\]
e.g.
(a) Show that \(x = 1\) is a root in \(f(x) = 3x^3 - x^2 - 10x + 8\)

(b) Hence or otherwise, factorise \(f(x)\) completely.

Solution:
(a) \( f(1) = 3(1)^3 - (1)^2 - 10(1) + 8 \)
\[ = 3 - 1 - 10 + 8 \]
\[ = 0 \]

(b) \( f(x) = (x - 1)(3x^2 + 2x - 8) \)
\[ = (x - 1)(3x - 4)(x + 2) \]

### 25.3 Rational Root Theorem

Let \( f(x) = ax^3 + bx^2 + cx + d \), if a rational root exists, it must be in the form of \( \frac{p}{q} \), where \( p \) divides \( d \), \( q \) divides \( a \), and \( p \) and \( q \) are coprime.

E.g. Completely factorise \( x^3 - x^2 - 14x + 24 \).

Solution:
Test \( x = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24 \), until \( f(x) = 0 \).
\( f(1) = 10 \)
\( f(-1) = -12 \)
\( f(2) = 0 \)

\( (x - 2) \) is a factor. Carry out polynomial division and further factorise. Steps omitted here.
\( f(x) = (x - 2)(x - 3)(x + 4) \)

E.g. Completely factorise \( f(x) = 2x^3 + 3x^2 + 7x + 3 \).

Solution
Test \( x \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2} \)
\( f(1) = 15 \)
\( f(-1) = -3 \)
\( f(2) = 45 \)
\[ f(-2) = -15 \]
\[ f\left(\frac{1}{2}\right) = \frac{15}{2} \]
\[ f\left(-\frac{1}{2}\right) = 0 \]

\( x + \frac{1}{2} \) is a factor, therefore \((2x + 1)\) is a factor.

\[ f(x) = (2x + 1)(x^2 + x + 3) \]

\[ f(x) = \begin{array}{c}
x^2 + x + 3 \\
2x + 1)2x^3 + 3x^2 + 7x + 3 \\
\hline
2x^3 + x^2 \\
2x^2 + 7x \\
\hline
2x^2 + x \\
6x + 3 \\
\hline
6x + 3 \\
0
\end{array} \]

\[ \Delta = 1 - 4(3) = -11 < 0 \]

Last revised on 13/5/2018 by Steve Cheung
26 Unit Vectors

Let, \[a = p\mathbf{i} + q\mathbf{j} = \begin{pmatrix} p \\ q \end{pmatrix}\quad b = r\mathbf{i} + s\mathbf{j} = \begin{pmatrix} r \\ s \end{pmatrix}\]

Addition \[a + b = (p + r)\mathbf{i} + (q + s)\mathbf{j} = \begin{pmatrix} p + r \\ q + s \end{pmatrix}\]

Subtraction \[a - b = (p - r)\mathbf{i} + (q - s)\mathbf{j} = \begin{pmatrix} p - r \\ q - s \end{pmatrix}\]

Multiplication by a scalar \[ca = c(p\mathbf{i} + q\mathbf{j})\]
\[= c\mathbf{i} +cq\mathbf{j} = \begin{pmatrix} cp \\ cq \end{pmatrix}\]

Length or Magnitude of a vector \[|a| = \sqrt{p^2 + q^2}\]

A unit vector of \(a\) \[\frac{a}{|a|} = \frac{1}{|a|}a\]

E.g. Suppose \(p = (4\mathbf{i} - \mathbf{j}), q = (\mathbf{i} - 2\mathbf{j})\),

(a) Find \(4\mathbf{p} - \mathbf{q}\).
(b) Find the magnitude of \(4\mathbf{p} - \mathbf{q}\).
(c) Hence or otherwise, find a unit vector for \(4\mathbf{p} - \mathbf{q}\).

Solution:
(a) \[4\mathbf{p} - \mathbf{q} = 4(4\mathbf{i} - \mathbf{j}) - (\mathbf{i} - 2\mathbf{j})\]
\[= 16\mathbf{i} - 4\mathbf{j} - \mathbf{i} + 2\mathbf{j}\]
\[= 15\mathbf{i} - 2\mathbf{j}\]

(b) \[|4\mathbf{p} - \mathbf{q}| = \sqrt{(15)^2 + (-2)^2}\]
\[= \sqrt{229}\]

(c) \[\frac{1}{\sqrt{229}}(15\mathbf{i} - 2\mathbf{j})\]
27 Vector Diagrams

If \( A, B \) have position vectors \( a, b \), respectively, then

\[
\overrightarrow{AB} = b - a
\]

This is a very useful result and is often asked.

If \( P \) is a point that divides \( AB \) internally by the ratio \( m : n \), then \( AP \) can be thought as travelling \( \frac{m}{m + n} \) of the way from \( A \) to \( B \), starting from \( A \). Therefore,

\[
\overrightarrow{AP} = \overrightarrow{OA} + \overrightarrow{AP}
\]

\[
= \overrightarrow{OA} + \left( \frac{m}{m + n} \right) \overrightarrow{AB}
\]

\[
= a + \frac{m}{m + n} (b - a)
\]

\[
= \frac{m + n}{m + n} a + \frac{m}{m + n} (b - a)
\]

\[
= \frac{ma + na + mb - ma}{m + n}
\]

\[
= \frac{na + mb}{m + n}
\]

\[
= \frac{n}{m + n} a + \frac{m}{m + n} b
\]

This is where the familiar form

\[
P = \left( \frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n} \right)
\]

comes from, where

\( x_1 = \text{i-component of } a, x_2 = \text{i-component of } b, y_1 = \text{j-component of } a, y_2 = \text{j-component of } b. \)
Two vectors are parallel if they can be expressed as scalar multiples of one another. In other words, 
\[ \overrightarrow{AB} \parallel \overrightarrow{CD} \] if

\[ \overrightarrow{AB} = k(\overrightarrow{CD}) \text{ or } \overrightarrow{CD} = l(\overrightarrow{AB}), \text{ where } k, l \in \mathbb{R} \]

In addition, \( PQR \) is a straight line (term: collinear) if any two vectors

\[ \overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{PR} \]

are parallel.

### 27.1 Ratios of Triangles

Most questions follow this basic structure. Quadrilaterals will be an exception. You are much better off finding the area of the shapes and simplify as needed.

Attempt to find a pair of equal angles and apply area of triangle using sine. These include same angle, vertically opposite angles, and supplementary angles. Remember that \( \sin(180 - x)^\circ = \sin x \).
In Fig. 5, $\overrightarrow{AB} = p$ and $\overrightarrow{AC} = 2q$. $D$ is the point on $BC$ such that $BD : DC = 1 : 3$ and $E$ is the mid-point of $AC$.

(a) Write down, in terms of $p$ and $q$, expressions for

(i) $\overrightarrow{BE}$,  
(ii) $\overrightarrow{BD}$.

(b) Show that $\overrightarrow{AD} = \frac{3}{4}p + \frac{1}{2}q$.

$AX = \lambda \overrightarrow{AD}$ and $BX = \mu \overrightarrow{BE}$, where $\lambda$ and $\mu$ are scalar constants.

By considering the triangle $BXA$

(c) find a relationship between $\lambda$, $\mu$, $p$ and $q$.

(d) Deduce the values of $\lambda$ and $\mu$.

(e) Write down the ratios

(i) area $\triangle BXA$ : area $\triangle BXD$,

(ii) area $\triangle BXA$ : area $\triangle EXA$,

(iii) area $\triangle BXD$ : area $\triangle EXA$. 
Solution:

(a) (i) \( \overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{AE} \)
\[
= -p + \frac{1}{2} \overrightarrow{AC}
= -p + q
= q - p
\]

(ii) \( \overrightarrow{BD} = \frac{1}{4} \overrightarrow{BC} \)
\[
= \frac{1}{4} (\overrightarrow{BA} + \overrightarrow{AC})
= \frac{1}{4} (-p + 2q)
= \frac{1}{4} (2q - p)
\]

(b) \( \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} \)
\[
= p + \frac{1}{4} (2q - p)
= p + \frac{1}{2} q - \frac{1}{4} p
= \frac{3}{4} p + \frac{1}{2} q
\]

c) \( \overrightarrow{BA} = \overrightarrow{BX} + \overrightarrow{XA} \)
\[
p = \mu \overrightarrow{BE} - \lambda \overrightarrow{BD}
= \mu (q - p) - \lambda (\frac{3}{4} p + \frac{1}{2} q)
= \mu q - \mu p - \frac{3}{4} \lambda p - \frac{1}{2} \lambda q
p = (-\mu - \frac{3}{4} \lambda) p + (\mu - \frac{1}{2} \lambda) q
\]

(d) \[-\mu - \frac{3}{4} \lambda = 1 \quad \mu = \frac{1}{2} \lambda\]
\[
(+) \quad \mu - \frac{1}{2} \lambda = 0
= \frac{2}{5} \left( \frac{4}{5} \right)
\]
\[-\frac{5}{4} \lambda = 1
= \frac{2}{5}
\]
\[
\lambda = \frac{4}{5}
\]
Labelling the diagram using $\mu$ and $\lambda$.

(e)

(i) area $\triangle BXA : \text{area} \triangle BXD$

$$\frac{1}{2}(XA)(XB) \sin BXA : \frac{1}{2}(XD)(XB) \sin BXD$$

$$= 4 \times 2 : 1 \times 2$$

$$= 4 : 1$$

(ii) area $\triangle BXA : \text{area} \triangle EXA$,

$$\frac{1}{2}(XB)(XA) \sin BXA : \frac{1}{2}(XE)(XA) \sin EXA$$

$$= 2 \times 4 : 3 \times 4$$

$$= 2 : 3$$

(iii) area $\triangle BXD : \text{area} \triangle EXA$

$$\frac{1}{2}(XB)(XD) \sin BXD : \frac{1}{2}(XE)(XA) \sin EXA$$

$$= 2 \times 1 : 3 \times 4$$

$$= 2 : 12$$

$$= 1 : 6$$

Note that in part (i) and (ii), the answers are obvious, since only one side of the triangle is changing if you compare the triangles to $\triangle BAD$.

If I ask you to find the ratio $\triangle BAE : \triangle BAC$, the angle used will be $BAC$ for both triangles.

Hopefully you gained some new insights from this. This was the example I showed in class which I think I explained differently.
28 Optimisation

Questions usually involve finding the maximum volume or minimum surface area if it is a 3D-shape, maximum area if it is 2D. You will need to know how to differentiate negative exponents in many cases.

Remember that the question usually asks for a minimum or maximum value, so plug in the critical value back into the volume or area expression.

Critical values come from setting $\frac{dA}{dx} = 0$ or $\frac{dV}{dx} = 0$, etc. If there are more than one, you might have to reject some (usually the negative value, since a dimension must be non-negative).

To show that it is a maximum or minimum, use the second derivative test.

- If $\left. \frac{d^2A}{dx^2} \right|_{x=p} > 0$, $A$ attains a minimum at $x = p$.

- If $\left. \frac{d^2V}{dx^2} \right|_{x=p} < 0$, $V$ attains a maximum at $x = p$, where $p$ is a critical value.

If second derivative test fails, which I have never encountered in this course, use first derivative test by drawing a sign diagram which states:

- If the first derivative switches from $+ve$ to $-ve$ at the critical value, then it is a maximum.

- If the first derivative switches from $-ve$ to $+ve$ at the critical value, then it is a minimum.
29  Related Rates

By chain rule,
\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}
\]

If we let \( x = y \), then
\[
\frac{dy}{dy} = \frac{dy}{du} \times \frac{du}{dy}
\]
\[
1 = \frac{dy}{du} \times \frac{du}{dy}
\]
\[
\frac{1}{dy} = \frac{du}{dy}
\]

This result is particularly important when we do related rates. This is like \( \log_a x = \frac{1}{\log_x a} \).

Many times, questions involve finding \( \frac{dr}{dt} \), which can be broken down as such:
\[
\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}
\]
\[
\frac{dr}{dt} = \frac{1}{dV} \times \frac{dV}{dr}
\]
\[
\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dt}{dr}
\]

You should know be familiar with the manipulations.

The objective of the question is usually finding the rate at a specific point. \( r \) may not be given directly. If given indirectly, solve a simple equation such as \( \pi r^2 = 36\pi \).

You can read the full solutions of related rates questions for more details.